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A Note on Effective Lagrangians in Matrix Theory

P. Berglund[1] and D. Minic[2,3]

[1] *Institute for Theoretical Physics*
University of California
Santa Barbara, CA 93106, USA

[2] *Enrico Fermi Institute*
University of Chicago
Chicago, IL 60637

[3] *Physics Department*
Penn State University
University Park, PA 16802

We study the relation between the effective Lagrangian in Matrix Theory and eleven dimensional supergravity. In particular, we provide a relationship between supergravity operators and the corresponding terms in the post-Newtonian approximation of Matrix Theory.

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Email: berglund@itp.ucsb.edu, dminic@yukawa.uchicago.edu

Matrix Theory [1], the conjectured non-perturbative formulation of the infinite-momentum limit of M-theory, even though formulated in a non-covariant, background dependent manner, seems to capture all the essential properties of string duality [2]. The comparison between Matrix Theory and its conjectured low-energy limit, eleven dimensional supergravity, in the limit of low velocities and large distances, has so far been remarkably successful [2]. One of the most amazing recent results is the fact that certain two-loop Matrix Theory effects are in complete numerical agreement with low energy supergravity calculations [3]. The authors of [3] developed a systematic double expansion in relative velocity and inverse separation, the so called post-Newtonian approximation, of the effective Lagrangian for higher order graviton-graviton scattering in Matrix Theory. This approach paves the way for further direct comparisons of Matrix Theory predictions and supergravity results. It is our aim in this note to elaborate on the general structure of effective Lagrangians in Matrix Theory in the post-Newtonian approximation, and further study its relationship with eleven dimensional supergravity effective Lagrangian [4,5]. We work in the framework of Susskind's discrete light-cone approach to Matrix Theory [6], as in [3], thus keeping the number N of D0-branes finite. (We take the light-cone coordinates to be defined as $x^{\pm} = t \pm x^{11}$, where $2x^{+} = \tau$, the light-cone quantization parameter, and x^{-} runs from 0 to $2\pi R$.) Furthermore, we point out that the scaling with N of another remarkable two-loop Matrix Theory effect, namely scattering of gravitons off $\mathbf{R}^8/\mathbf{Z}_2$ orbifold points [7], agrees with eleven dimensional supergravity.

Let us first briefly review the simple systematics of effective Matrix Theory Lagrangians, following [3]. Our starting point is the fundamental infinite-momentum frame Matrix Theory Lagrangian [1] (for earlier work on the $N = 16$ supersymmetric quantum mechanics, see [8])

$$S = \int dt(\frac{1}{2R} D_t X_i D_t X_i + \bar{\Psi} D_t \Psi + \frac{1}{4} M^6 R [X_i, X_j]^2 - M^3 R \bar{\Psi} \gamma^i [X_i, \Psi]) \,, \tag{1}$$

where X_i are nine $N \times N$ matrices (N being kept finite), accompanied by sixteen supersymmetric partners Ψ, R is the extent of the eleventh dimension, and M is the eleven dimensional Planck mass. After rescaling $tR = \tau$, $M^3 X_i = y_i$, $M^3 \Psi = \psi$,

$$S = M^{-6} \int d\tau(\frac{1}{2} D_\tau y_i D_\tau y_i + \bar{\psi} D_\tau \psi + \frac{1}{4}[y_i, y_j]^2 - \bar{\psi}\gamma^i [y_i, \psi]) \,. \tag{2}$$

Thus, M^6 is the appropriate loop expansion parameter. The effective loop action then becomes

$$S_l = M^{6l-6} \int d\tau f_l(y_i, \psi, D_\tau) = M^{6l-6} R \int dt f_l(M^3 X_i, M^3 \Psi, R^{-1} D_t) \,. \tag{3}$$

where f_l is some undetermined function. On dimensional grounds f_l has to scale as M^{8-6l}; therefore

$$S_l = M^{6l-6} R \int dt (M^3 r)^{4-3l} f_l(\frac{X_i}{r}, \frac{\Psi}{r}, \frac{D_t X_i}{RM^3 r^2}) , \tag{4}$$

with $r^2 = X_i X_i$ and $v_i = D_t X_i$. The effective loop Lagrangian at loop order n, L_n, written as a function of $\frac{v^2}{R^2 M^6 r^4}$ (the powers of v have to be even due to time-reversal invariance [3]) then reads as follows

$$L_n = \sum_{m=0}^{\infty} \frac{v^2}{R} c_{nm} (\frac{1}{M^3 r^3})^n (\frac{v^2}{R^2 r^4 M^6})^m = \sum_{m=0}^{\infty} \frac{v^2}{R} c_{nm} (\frac{1}{r^3})^n (\frac{v^2}{r^4})^m (\frac{1}{R^2})^m (\frac{1}{M^3})^{n+2m} , \tag{5}$$

or

$$L_n = \sum_{m=0}^{\infty} \frac{v^2}{R} c_{nm} (\frac{v^2}{r^7})^n (\frac{v^2}{r^4})^{m-n} (\frac{1}{R^2})^m (\frac{1}{M^3})^{n+2m} . \tag{5a}$$

Here c_{nm} denotes the coefficient of the mth term at the nth loop order. Note that one can deduce the following explicit formula from the classic result of [9],

$$c_{1n} = \frac{-2\Gamma(4n+2)}{\Gamma(2n+1)\Gamma(2n+3)} \frac{(-1)^{n+1} 2(1-2^{2n+2}) B_{n+1} - (n+1)}{2^{4n}} , \tag{6}$$

where B_n stands for the Bernoulli numbers, $B_n = \frac{\Gamma(2n+1)}{\pi^{2n} 2^{2n-1}} \sum_{k=1}^{k=\infty} \frac{1}{k^{2n}}$. Similarly, the tour de force calculations of [10] and [7] allow one to deduce in principle c_{2n}, albeit, apparently, not in a closed form.

A very interesting physical limit of (5a) is obtained for $n = m$

$$L_{diag} = \sum_n c_{nn} \frac{v^2}{R} (\frac{v^2}{R^2 M^9 r^7})^n . \tag{7}$$

This Lagrangian can be generated by considering the action of a probe graviton in the background of a classical source [3]. Note that the explicit powers of N are not included in the above formula. In order to do that, one has to apply Susskind's discrete light-cone procedure according to which the light like momentum is positive, finite and quantized as $p_- = N/R$ [3,6]. (The light like compactification of [6], is to be understood as a limit of a space-like compactification, when the space-like vector becomes null [3,11].) More explicitly, the expansion parameter in (7) comes from the action for a massless probe in the Aichelburg-Sexl background in the discrete light-cone frame (the only component of the metric induced by the classical source being $h_{--} = \frac{15 N_s \pi}{2R^2 M^9 r^7}$) . Then (7) can be rewritten as

$$L_{diag} = -p_- \dot{x}^- , \tag{8}$$

where $\dot{x}^- = \frac{\sqrt{1-h_{--}v^2}-1}{h_{--}}$, which follows from the classical equation of motion. Therefore $c_{nn} = (-1)^{n+2}\binom{\frac{1}{2}}{n+1}(15/2)^n$ (for a given number of loops n in the Matrix Theory effective Lagrangian).

This result implies the following important selection rules,

$$\Delta P(N_s) = 1\,, \quad \Delta P(N_p) = 0\,, \quad \Delta n = 1\,, \tag{9a}$$

$$\Delta P(N_s) = 0\,, \quad \Delta P(N_p) = 0\,, \quad \Delta n = 0\,, \tag{9b}$$

where $N_{s,p}$ denote the number of source/probe $D0$-branes, and $P(N_{s,p})$ powers of $N_{s,p}$). In other words, powers of N_s come from the metric only, while powers of N_p come from p_- of the probe. (The nth loop effective Lagrangian scales as $N_s^n N_p$, which agrees with the N dependence in the double-line prescription for diagrams with n loops.)

Armed with the above selection rule, we proceed to study the general form of the effective Lagrangian (5) and (5a). First we note that the general term in (5a) can be generated by considering two types of operations, starting with the well-known $\frac{v^4}{r^7}$ term: a) horizontal "moves", which contribute $(m-n)$ powers of $\frac{v^2}{r^4}$, and b) diagonal "moves", which give n powers of $\frac{v^2}{r^7}$. The two operations can be deduced by specifying the counting of powers of v and r. To do so, we observe that in Susskind's discrete light-cone formulation of Matrix Theory [6],

$$\partial_+ \sim v^2, \quad \partial_i \sim v, \quad h_{--} \sim \frac{N_s}{R^2 M^9 r^7}, \quad p_- \sim N_p/R\,. \tag{10}$$

For example, $\frac{v^4}{r^7}$ corresponds to $(h_{--})_s(\partial_+\partial_+)_p$, which in turn corresponds to $(h_{\mu\nu})_s(\mathcal{R}_{\mu\nu})_p$. (Here we have separated operators associated with the source, from the operators associated with the probe.)

Given the counting (10) and the selection rules (9), one can construct the following horizontal and diagonal operations,

$$\text{Horizontal(H)} \quad \rightarrow \quad (\partial^2)_s^2(\partial_i\partial_i)_p\,, \tag{11a}$$

$$\text{Diagonal(D)} \quad \rightarrow \quad (h_{--})_s\Big(\frac{\partial_+\partial_+}{\partial_i\partial_i}\Big)_p\,. \tag{11b}$$

The form of the diagonal operator is fixed by the fact that the source provides the classical background field and by the selection rule which states that the power of N_s can only change by one, after applying (11b). This in turn implies that each diagonal move can have only one power of the metric and therefore, due to the fact that the indices of the background metric h_{--} can be contracted only by the action of ∂_+, associated with the probe (and suitably normalized so to get the right powers of velocity), each diagonal move has to contain two powers of ∂_+. The form of the horizontal operator follows from the

fact that the "nonlocal" part of the diagonal operator has to be canceled in order to get an overall local operator acting on the probe. Also the horizontal move cannot contain powers of the metric because of the selection rule which states that the power of N_s does not change, after an application of a horizontal move. Thus, different operators are obtained by applying, symbolically, $D^n H^{m-n}$ on the first term in the expansion (5a).

Note that we have tacitly assumed that any composite operator $\mathcal{O}$ should be represented in the post-Newtonian approximation as $\mathcal{O}_{1s}\mathcal{O}_{2p}$. This implies that powers of velocity come only from operators acting on the probe and powers of inverse distance come only from operators acting on the source, while powers of N_s are generated only by the diagonal moves.

Now we use (11) in order to construct operators that appear in the effective eleven dimensional supergravity Lagrangian as in [5]. For example, let us consider the operator $\mathcal{R}^{2i} \to \mathcal{R}_s^i \mathcal{R}_p^i$. This operator is generated by the (n,m)th term in the effective Matrix Theory Lagrangian (5a), for $i = 3(n-1) - 1$ and $m = 2i - 1$. In other words, the $\mathcal{R}^{2i}$ operator in supergravity corresponds to the n-th loop $\frac{v^{2+2m}}{r^{3n+4m}} = \frac{v^{4i}}{r^{9i}}$ term in Matrix Theory. In particular for the $\mathcal{R}^4$ term discussed in [5], $m = 3$, $n = 2$, so the $\mathcal{R}^4$ operator in supergravity corresponds to the two-loop $\frac{v^8}{r^{18}}$ term in Matrix Theory. Given the results of [10], we see that Matrix Theory provides an explicit prediction of the coefficient of the $\mathcal{R}^4$ term [12]. Note that this term is proportional to N_s^2, in agreement with the general selection rule. Note also, that operators such as $\mathcal{R}^2$ and $\mathcal{R}^6$ do not correspond to any terms in the effective Matrix Theory Lagrangian, which is consistent with the results of [5].

Turning to the operator $\mathcal{R}^{2i+1}$ one can similarly show that $i = 3(n-1)$ and $m = 2i+1$. Then, for example, $\mathcal{R}^3$ and $\mathcal{R}^5$ do not correspond to any terms in the Matrix Theory effective Lagrangian. Thus, we find that only operators of the form $\mathcal{R}^{3m-1}$, where $m = 1, 2, ...$, are allowed, which is in perfect agreement with [5]. Furthermore, from (11a), we find that the operator creating the horizontal move has dimension $-2 \cdot 3$. We can then generalize what we mean by $\mathcal{R}^k$ (as was done in [5]) to also include covariant derivatives as well as scalars made from $\mathcal{R}$, as long as the dimension is $-2 \cdot k$; the statement that only operators of the form $\mathcal{R}^{3m-1}$ are allowed still holds.

One curious fact that stems from the general formula for the coefficients in the one-loop Matrix Theory Lagrangian, is that only the second coefficient is zero. (The other coefficients are non-zero, which can be proven from general properties of the Bernoulli numbers). This particular term turns out to correspond (in accordance with our general rules) to $(\partial^2\mathcal{R})_s(\partial_k^2\mathcal{R})_p$. By applying one diagonal move on this operator and dropping the total-derivative terms, the $\mathcal{R}^4$ term is obtained as it should.

We emphasize that the above outlined procedure cannot say anything about numerical coefficients in front of the generated operators.

Finally, we can apply our general counting (in particular the counting of powers of N) to the problem considered in [7], namely the scattering of a graviton off a $\mathbf{R}_8/\mathbf{Z}_2$

orbifold point. We would like to point out that there exists no disagreement in scaling of the effective Lagrangian for this process with N, between Matrix Theory (formulated in discrete light-cone gauge) and eleven dimensional supergravity.

The two-loop Matrix Theory result (the gauge group is $U(N) \times U(N)$) for scattering of a graviton off a $\mathbf{R}_8/\mathbf{Z}_2$ fixed point is, according to [7],

$$L_{orbifold} \sim \frac{N^3 v^2}{r^6} \,. \tag{12}$$

Note that in this case we have only one N, because only one graviton scatters off a fixed point. (Hence, there are no symmetrization factors in the Feynman diagrams as in $\mathbf{R}^9$ [3].) This result agrees with the general selection rules (9) and the expression (5a).

In the case of supergravity the authors of [7] considered the scattering of a graviton probe off a membrane, which couples to the three-form C_3 in eleven dimensional supergravity. (Any process involving scattering of gravitons off each other, e.g. scattering of a graviton off its Z_2 mirror, would necessarily go at least as v^4, one power of v^2 for each graviton). The classical solution representing a membrane in eleven dimensional supergravity is given by the following expression, as in [13]

$$\begin{aligned} ds^2 &= H^{-2/3}(-dt^2 + dy_1^2 + dy_{11}^2) + H^{1/3} dx_i dx^i \\ C_3 &= H^{-1} dt \wedge dy_1 \wedge dy_{11} \\ H &= 1 + \frac{Q}{r^6} \end{aligned} \tag{13}$$

where $Q \sim N$, and N is the number of $D0$-branes (see below). Then the supergravity potential can be read off from the geodesic equation (or from the tree-level Feynman diagram for this process) [7]

$$V(r) \sim \frac{N^3 v^2}{r^6} \,. \tag{14}$$

Here, one power of N comes from the light like momentum of the probe while N^2 comes from the membrane. We treat the membrane as a bound state of N $D0$-branes, and therefore set the charge Q proportional to N. Another power of N comes from the energy momentum tensor of the source which is proportional to the light like momentum of the membrane. Thus the powers of N agree explicitly as in the case of [3]. This, in our view, demonstrates one more time the usefulness of the discrete light-cone approach to Matrix Theory.

Note that the $\frac{v^2}{r^6}$ term appears at second loop in the effective Lagrangian (5a). In fact, this is the very term that can be obtained by two inverse horizontal moves from the second term on the diagonal (the familiar $\frac{v^6}{r^{14}}$ term). The overall powers of N agree according to (11).

We also remark that in case the orbifold point is $\mathbf{R}^5/\mathbf{Z}_2$, the one loop Matrix Theory result (the gauge group is $Sp(N)$), which scales as $\frac{Nv^2}{r^3}$ [14,7] agrees with the corresponding supergravity result (in particular the scaling with N) because the energy of a longitudinal five-brane is constant in the infinite momentum frame [15].

Finally, one could try to repeat the argument presented in [3] for the metric given by (13) and generate higher order terms in the effective Lagrangian, starting from the two-loop term $\frac{N^3v^2}{r^6}$. By applying the diagonal operation, which scales as $\frac{N^3v^2}{r^6}$, since the source is the eleven dimensional membrane, one would obtain terms of the form $\frac{N^{3+l}v^{2+2l}}{r^{6+6l}}$. However, such terms would not be allowed in the general form of (5a). In particular, the powers of inverse distance would not match the expansion (5a). Furthermore, by taking the horizontal moves in the opposite direction, one can in a similar way show that the higher loop corrections to the $\frac{N^3v^2}{r^6}$ are not of the form (5a). Also, each loop introduces another power of r^{-3}, and is subleading. Thus, we conclude that the two-loop term does not get further corrected (this was noticed in [7]). (Similar arguments can be applied to the situation recently described in [16]. There it is argued that the one-loop corrected v^2 potential for scattering of zero branes off elementary string in Matrix Theory is exact.)

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References

[1] T. Banks, W. Fischler, S. Shenker and L. Susskind, *M Theory As A Matrix Model: A Conjecture*, Phys. Rev. D**55** (1997) 5112, hep-th/9610043.

[2] For a review, see T. Banks, *The State of Matrix Theory*, hep-th/9706168.

[3] K. Becker, M. Becker, J. Polchinski and A. Tseytlin, *Higher Order Graviton Scattering in M(atrix) Theory*, hep-th/9706072.

[4] M. B. Green and P. Vanhove, *D-instantons, Strings and M-theory*, hep-th/9704145; M. B. Green, M. Gutperle, P. Vanhove, *One loop in eleven dimensions*, hep-th/9706175.

[5] J. Russo and A. A. Tseytlin, *One-loop four-graviton amplitude in eleven-dimensional supergravity*, hep-th/9707134.

[6] L. Susskind, *Another Conjecture about M(atrix) Theory*, hep-th/9704080.

[7] O. J. Ganor, R. Gopakumar and S. Ramgoolam, *Higher Loop Effects in M(atrix) Orbifolds*, hep-th/9705188.

[8] M. Claudson and M. Halpern, *Supersymmetric Ground State Wave Functions*, Nucl. Phys. B250 (1985) 689;
M. Baake, M. Reinicke and V. Rittenberg, *Fierz Identities For Real Clifford Algebras And The Number of Supercharges*, J. Math. Phys. 26 (1985) 1070;
R. Flume, *On Quantum Mechanics With Extended Supersymmetry And Nonabelian Gauge Constraints*, Ann. Phys. 164 (1985) 189.

[9] C. Bachas, *D-brane dynamics*, Phys. Lett. 374 B (1996) 37, hep-th/9511043;
M. R. Douglas, D. Kabat, P. Pouliot and S. H. Shenker, *D-branes and Short Distances in String Theory*, Nucl. Phys. B485 (1997) 85, hep-th/9608024.

[10] K. Becker and M. Becker, *A Two-Loop Test of M(atrix) Theory*, hep-th/9705091.

[11] S. Hellerman and J. Polchinski, work in progress.

[12] K. Becker, M. Becker and M. B. Green, work in progress.

[13] J. Russo, *BPS Bound States, Supermembranes and T-Duality in M-Theory*, hep-th/9703118.

[14] N. Kim and S.-J. Rey, *M(atrix) Theory on T_5/Z_2 Orbifold and Five-Brane*, hep-th/9705132.

[15] T. Banks, N. Seiberg and S. Shenker, *Branes from Matrices*, hep-th/9612157.

[16] R. Gopakumar and S. Ramgoolam, *Scattering of zero branes off elementary strings in Matrix theory*, hep-th/9708022.